

N.B:- (1) Question no. 1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1.(a) Evaluate $\int_0^{\infty} 3^{-4x^2} dx$ [3]

Ans: Let $I = \int_0^{\infty} 3^{-4x^2} dx$

put $3^{-4x^2} = e^{-t}$

taking log on both sides,

$$4x^2 \log 3 = t$$

$$x^2 = \frac{t}{4 \log 3} \Rightarrow x = \frac{\sqrt{t}}{2\sqrt{\log 3}}$$

diff. w.r.t x,

$$dx = \frac{t^{-1/2}}{4\sqrt{\log 3}} dt \quad \text{lim} \rightarrow [0, \infty]$$

$$\therefore I = \int_0^{\infty} \frac{e^{-t}}{4\sqrt{\log 3}} t^{-1/2} dt$$

$$\therefore I = \frac{1}{4\sqrt{\log 3}} \int_0^{\infty} e^{-t} \cdot t^{-1/2} dt$$

$\therefore I = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}${ $\int_0^{\infty} e^{-t} \cdot t^{-1/2} dt = \sqrt{\pi}$ }
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(b) Solve $(2y^2 - 4x + 5)dx = (y - 2y^2 - 4xy)dy$ [3]

Ans : $(2y^2 - 4x + 5)dx = (y - 2y^2 - 4xy)dy$

Compare with $Mdx + Ndy = 0$

$$\therefore M = (2y^2 - 4x + 5) \quad \therefore N = -(y - 2y^2 - 4xy)$$

$$\frac{\partial M}{\partial y} = 4y$$

$$\frac{\partial N}{\partial x} = 4y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given diff. eqn is exact .

The solution of exact diff. eqn is given by ,

$$\int M dx + \int [N - \frac{\partial}{\partial y} M dx] dy = c$$

$$\int M dx = \int ((2y^2 - 4x + 5)) dx = 2xy^2 - 2x^2 + 5x$$

$$\frac{\partial}{\partial y} \int M dx = 4xy$$

$$\int [N - \frac{\partial}{\partial y} M dx] dy = \int [4xy - y + 2y^2 - 4xy] dy = \frac{2}{3} y^3 - \frac{y^2}{2}$$

$$\therefore 2xy^2 - 2x^2 + 5x + \frac{2}{3} y^3 - \frac{y^2}{2} = c$$

(c) Solve the ODE $(D - 1)^2(D^2 + 1)^2 y = 0$ [3]

Ans : $(D - 1)^2(D^2 + 1)^2 y = 0$

For complementary solution ,

$$f(D) = 0$$

$$(D - 1)^2(D^2 + 1)^2 = 0$$

$$\therefore (D - 1)^2 = 0 \quad \therefore (D^2 + 1)^2 = 0$$

$$D - 1 = 0 \text{ for two times} \quad (D^2 + 1) = 0 \text{ for two times}$$

$$\therefore D - 1 = 0 \quad \therefore D^2 = -1$$

Roots are : $D = 1, 1, +i, +i, -i, -i$

$$\therefore y_c = (c_1 + xc_2)e^x + [(c_3 + xc_4)\cos x + (c_5 + xc_6)\sin x]$$

(d) Evaluate $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}} dy dx$

[3]

Ans: let $I = \int_0^1 \int_0^{x^2} e^{\frac{y}{x}} dy dx$

$$= \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}} \right]_0^{x^2} dx$$
$$= \int_0^1 \frac{(e^{x^2} - 1)}{\frac{1}{x}} dx$$
$$= \int_0^1 x \cdot e^{x^2} dx - \int_0^1 x \cdot dx$$
$$= [x \cdot e^{x^2} - e^{x^2}]_0^1 - \left[\frac{x^2}{2} \right]_0^1$$
$$= e - e + 1 - \frac{1}{2}$$

$$\therefore I = \frac{1}{2}$$

(e) Evaluate $\int_0^1 \frac{x^a - 1}{\log x} dx$

[4]

Ans: let $I = \int_0^1 \frac{x^a - 1}{\log x} dx$

Taking 'a' as parameter ,

$$I(a) = \int_0^1 \frac{x^a - 1}{\log x} dx \quad \text{----- (1)}$$

differentiate w.r.t a ,

$$\frac{dI(a)}{da} = \frac{d}{da} \int_0^1 \frac{x^a - 1}{\log x} dx$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{\partial}{\partial a} \frac{x^a - 1}{\log x} dx \quad \text{.....\{D.U.I.S f(x)\}}$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{x^a \cdot \log x}{\log x} dx \quad \text{.....\{ \frac{dx^a}{da} = x^a \cdot \log a \}}$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 x^a dx$$

$$\therefore \frac{dI(a)}{da} = \left[\frac{x^{a+1}}{a+1} \right]_0^1$$

$$\therefore \frac{dI(a)}{da} = \frac{1}{a+1} - 0$$

$$\therefore \frac{dI(a)}{da} = \frac{1}{a+1}$$

now , integrate w.r.t a,

$$I(a) = \int \frac{1}{a+1} da$$

$$I(a) = \log(a+1) + c \quad \text{----- (2)}$$

where c is constant of integration

put a=0 in eqn (1),

$$I(0) = \int_0^1 0 dx = 0$$

And

From eqn (2), $I(0) = c$

$$\therefore c = 0$$

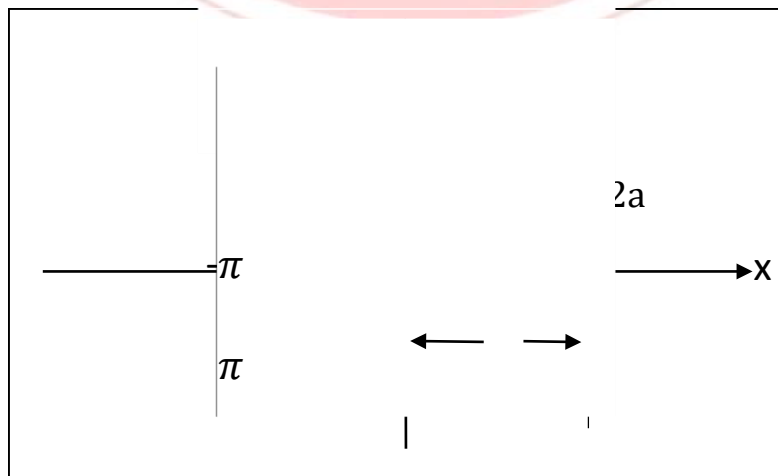
$$\therefore I = \log(a+1)$$

(f) Find the length of cycloid from one cusp to the next , where

$$x=a(\theta + \sin \theta), \quad y=a(1-\cos \theta).$$

[4]

Ans : Given curve : Cycloid $x=a(\theta + \sin \theta), \quad y=a(1-\cos \theta)$



The length of given curve is :

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$\frac{dx}{d\theta} = a(1 + \cos \theta) \quad \frac{dy}{d\theta} = a \sin \theta$$

$$\begin{aligned} \therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= a^2 [1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta] \\ &= 2a^2 [1 + \cos \theta] \\ &= 4a^2 [\cos^2 \theta / 2] \end{aligned}$$

$$\therefore \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2a \cos \theta / 2$$

$$\begin{aligned} \therefore S &= \int_{-\pi}^{\pi} 2a \cos \theta / 2 d\theta \\ &= 2 \times \int_0^{\pi} 2a \cos \theta / 2 d\theta \\ &= 4a [2 \sin \theta / 2]_0^{\pi} \end{aligned}$$

$$\therefore S = 8a$$

Q.2.(a) Solve $(D^2 - 3D + 2)y = 2e^x \sin\left(\frac{x}{2}\right)$

[6]

Ans : $(D^2 - 3D + 2)y = 2e^x \sin\left(\frac{x}{2}\right)$

For complementary function ,

$$f(D) = 0$$

$$\therefore (D^2 - 3D + 2) = 0$$

Roots are : $D = 2, 1$ Real roots .

$$y_c = c_1 e^x + c_2 e^{2x}$$

For particular integral ,

$$\begin{aligned}
 y_p &= \frac{1}{f(D)} X \\
 &= \frac{1}{(D^2-3D+2)} 2e^x \sin\left(\frac{x}{2}\right) \\
 &= 2e^x \frac{1}{(D+1)^2-3(D+1)+2} \sin\left(\frac{x}{2}\right) \\
 &= 2e^x \frac{1}{(D^2-D)} \sin\left(\frac{x}{2}\right) \\
 &= 2e^x \frac{1}{-\left(\frac{1}{4}\right)-D} \sin\left(\frac{x}{2}\right) \\
 &= -8e^x \frac{1}{4D+1} \sin\left(\frac{x}{2}\right) \\
 &= -8e^x \frac{4D-1}{16D^2-1} \sin\left(\frac{x}{2}\right)
 \end{aligned}$$

$$y_p = \frac{8}{5} e^x \left(-\sin\left(\frac{x}{2}\right) - 2\cos\left(\frac{x}{2}\right) \right)$$

The general solution of given diff. eqn is given by,

$$y_c = y_c + y_p = c_1 e^x + c_2 e^{2x} + \frac{8}{5} e^x \left(-\sin\left(\frac{x}{2}\right) - 2\cos\left(\frac{x}{2}\right) \right)$$

(b) Using D.U.I.S prove that $\int_0^\infty e^{-(x^2+\frac{a^2}{x^2})} dx = \frac{\sqrt{\pi}}{2} e^{-2a}, a > 0$ [6]

Ans : Let $I(a) = \int_0^\infty e^{-(x^2+\frac{a^2}{x^2})} dx$ (1)

Taking 'a' as parameter diff. w.r.t. a,

$$\frac{dI(a)}{da} = \frac{d}{da} \int_0^\infty e^{-(x^2+\frac{a^2}{x^2})} dx$$

Apply D.U.I.S rule ,

$$\frac{dI(a)}{da} = \int_0^\infty \frac{\partial}{\partial a} e^{-(x^2+\frac{a^2}{x^2})} dx$$

$$= \int_0^\infty e^{-(x^2 + \frac{a^2}{x^2})} \cdot \frac{-2a}{x^2} dx$$

Put $\frac{a}{x} = t$, $\frac{-a}{x^2} dx = dt$

Limits [$\infty, 0$]

$$\frac{dI(a)}{da} = \int_\infty^0 e^{-(t^2 + \frac{a^2}{t^2})} \cdot 2dt = -2 \int_0^\infty e^{-(t^2 + \frac{a^2}{t^2})} dt = -2I(a)$$

$$\frac{dI(a)}{da} = -2I(a)$$

$$\therefore \frac{dI(a)}{I(a)} = -2da$$

Integrating both sides ,

$$\log [I(a)] = -2a + \log c$$

$$I(a) = c \cdot e^{-2a}$$

put $a=0$ in above eqn and eqn (1)

$$\therefore I(a) = c = \int_0^\infty e^{-x^2} dx = \frac{\pi}{\sqrt{2}} \dots\dots\dots \{ \text{Using gamma function} \}$$

$$\therefore I(a) = \frac{\pi}{\sqrt{2}} e^{-2a}$$

(c) Change the order of integration and evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x \, dx \, dy}{\sqrt{x^2+y^2}}$ [8]

Ans : Let $I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x \, dx \, dy}{\sqrt{x^2+y^2}}$

Region of integration is : $x \leq y \leq \sqrt{2-x^2}$

$$0 \leq x \leq 1$$

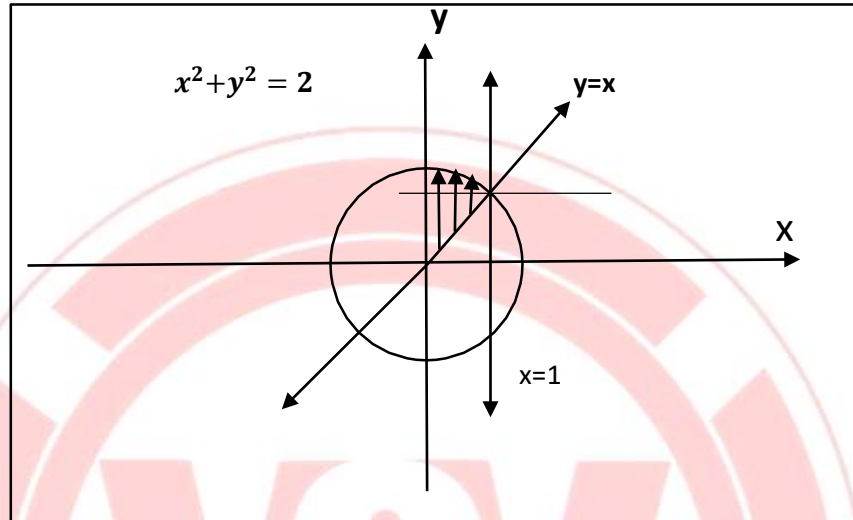
Curves : (i) $y = x$ line

(ii) $x=0, x=1$ lines parallel to the y axis .

$$(iii) y = \sqrt{2 - x^2} \Rightarrow x^2 + y^2 = 2$$

Circle with centre (0,0) and radius $\sqrt{2}$.

Intersection of circle and $y = x$ line is (1,1) in 1st quadrant.



Divide the region into two parts as shown in fig.

After changing the order of integration :

For one region : $0 \leq x \leq y$

$$0 \leq y \leq 1$$

For another region : $0 \leq x \leq \sqrt{2 - y^2}$

$$1 \leq y \leq \sqrt{2}$$

$$\begin{aligned} \therefore I &= \int_0^1 \int_0^y \frac{x \, dx \, dy}{\sqrt{x^2 + y^2}} + \int_1^{\sqrt{2}} \int_0^{\sqrt{2 - y^2}} \frac{x \, dx \, dy}{\sqrt{x^2 + y^2}} \\ &= \int_0^1 [\sqrt{x^2 + y^2}]_0^y \, dy + \int_1^{\sqrt{2}} [\sqrt{x^2 + y^2}]_0^{\sqrt{2 - y^2}} \, dy \\ &= \int_0^1 (\sqrt{2} \cdot y - y) \, dy + \int_1^{\sqrt{2}} (\sqrt{2} - 1) \, dy \\ &= (\sqrt{2} - 1) \left[\frac{y^2}{2} \right]_0^1 + [\sqrt{2}y - \frac{y^2}{2}]_1^{\sqrt{2}} \end{aligned}$$

$$= 1 - \frac{1}{\sqrt{2}}$$

$$\therefore I = \frac{\sqrt{2}-1}{\sqrt{2}}$$

Q.3(a) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dx dy dz$ [6]

Ans : Let $I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dx dy dz$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dz dy dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{1}{-2(x+y+z+1)^2} \right]_0^{1-x-y} dy dx$$

$$= - \int_0^1 \int_0^{1-x} \frac{1}{2} \left[\frac{1}{(x+y+1-x-y+1)^2} - \frac{1}{(x+y+1)^2} \right] dy dx$$

$$= - \int_0^1 \frac{1}{2} \left[\frac{1}{4} y + \frac{1}{(x+y+1)^1} \right]_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{2} \left\{ \left[\frac{1}{4} (1-x) - \frac{1}{2} \right] + \left[\frac{1}{x+1} \right] \right\} dx$$

$$= \frac{1}{2} \left[\frac{1}{4} \left(\frac{(1-x)^2}{8} \right) - \frac{x}{2} + \log(x+1) \right]_0^1$$

$$\therefore I = \frac{1}{2} \left[\log 2 - \frac{5}{8} \right]$$

(b) Find the mass of the lemniscate $r^2 = a^2 \cos 2\theta$ if the density at any point is Proportional to the square of the distance from the pole . [6]

Ans : Given curve : $r^2 = a^2 \cos 2\theta$ is lemniscate.

The density at any point is proportional to the square of dist. From the pole.

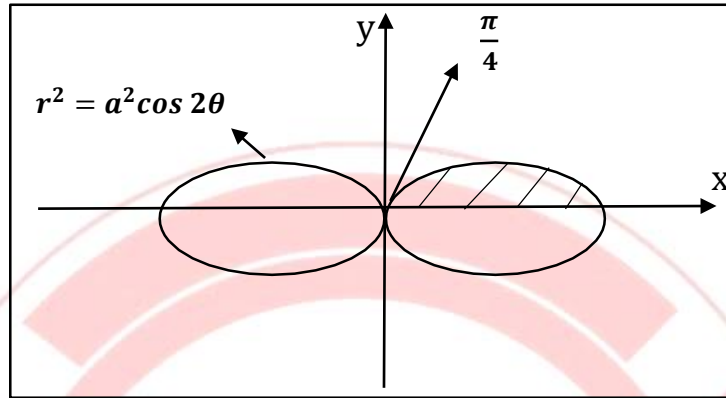
Distance from the pole = r

$$\therefore \text{Density} \propto r^2$$

$$\therefore \text{Density} = k.r^2$$

The mass of the lemniscate is given by ,

$$M = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \text{density } r \, dr \, d\theta$$



$$\begin{aligned} \therefore M &= 4 \times \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} k \cdot r^2 \cdot r \, dr \, d\theta \\ &= 4k \times \int_0^{\frac{\pi}{4}} \left[\frac{r^4}{4} \right]_0^{a\sqrt{\cos 2\theta}} d\theta \\ &= k \times \int_0^{\frac{\pi}{4}} a^4 \cdot \cos^2 2\theta \cdot d\theta \end{aligned}$$

We can solve this definite integral by beta function.

$$\text{Put } 2\theta = t \Rightarrow 2 \, d\theta = dt$$

$$\text{Limits } \left[0, \frac{\pi}{2} \right]$$

$$\begin{aligned} \therefore M &= ka^4 \int_0^{\frac{\pi}{2}} \cos^2 t \cdot \frac{dt}{2} \\ &= \frac{ka^4}{2} \times \frac{1}{2} \beta \left(\frac{1}{2}, \frac{3}{2} \right) \end{aligned}$$

$$\boxed{\therefore M = \frac{ka^4\pi}{8}}$$

(c) Solve $x^2 \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = 4 \log x$

[8]

Ans : $x^2 \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = 4 \log x$

The given diff. eqn is Cauchy's homogeneous eqn .

Multiply the given eqn by x,

$$x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 4x \log x$$

Put $x = e^z$ $\log x = z$

Diff. w.r.t x,

$$\frac{1}{x} = \frac{dz}{dx} \quad \text{but} \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x}$$

$$\therefore x \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y \quad \text{where } D = \frac{d}{dz}$$

$$\therefore [D(D-1)(D-2) + 3D(D-1) + D + 1]y = 4z \cdot e^z$$

$$\therefore [D^3 + 1]y = 4z \cdot e^z$$

For complementary solution ,

$$f(D) = 0$$

$$\therefore [D^3 + 1] = 0$$

$$\text{Roots are: } D = -1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Roots of the eqn are real and complex.

$$\therefore y_c = c_1 e^{-z} + e^{z/2} \left(c_2 \cos \frac{\sqrt{3}z}{2} + c_3 \sin \frac{\sqrt{3}z}{2} \right)$$

For particular integral ,

$$y_p = \frac{1}{f(D)} X = \frac{1}{(D^3+1)} 4z \cdot e^z$$

$$= 4e^z \frac{1}{(D+1)^3+1} z$$

$$= 4e^z \frac{1}{D^3+3D^2+3D+2} z$$

$$\therefore y_p = e^z(2z - 3)$$

The general solution of given diff. eqn is ,

$$y_g = y_c + y_p = c_1 e^{-z} + e^{z/2} \left(c_2 \cos \frac{\sqrt{3}z}{2} + c_3 \sin \frac{\sqrt{3}z}{2} \right) + e^z(2z - 3)$$

Resubstitute z,

$$\therefore y_g = \frac{c_1}{x} + \sqrt{x} \left(c_2 \cos \frac{\sqrt{3} \log x}{2} + c_3 \sin \frac{\sqrt{3} \log x}{2} \right) + x(2 \log x - 3)$$

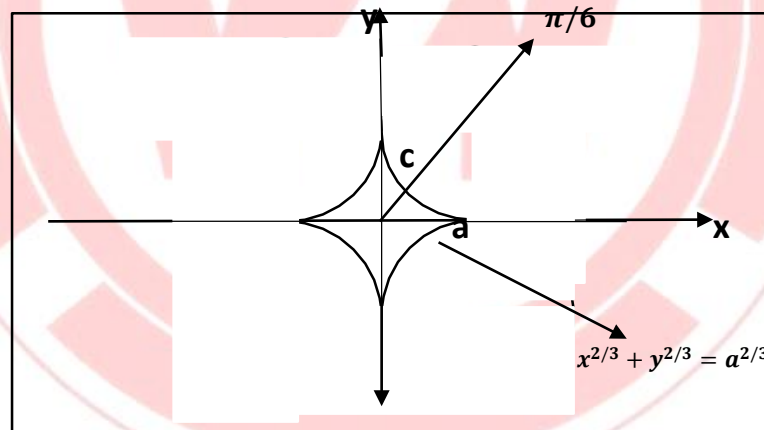
Q.4(a) Prove that for an astroid $x^{2/3} + y^{2/3} = a^{2/3}$, the line $\theta = \pi/6$

Divide the arc in the first quadrant in a ratio 1:3.

[6]

Ans : Given curve : astroid $x^{2/3} + y^{2/3} = a^{2/3}$

The line $\theta = \pi/6$ cuts the astroid in 1st quadrant.



C is the point on the curve which cuts the arc.

Length of astroid in first quadrant:

Put $x = a \cos^3 t$ and $y = a \sin^3 t$

$$dx = -3a \sin t \cdot \cos^2 t dt \quad dy = 3a \cos t \cdot \sin^2 t dt$$

$$S = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{\pi/2} \sqrt{(-3a \sin t \cdot \cos^2 t)^2 + (3a \cos t \cdot \sin^2 t)^2} dt$$

$$\begin{aligned}
 &= \int_0^{\pi/2} 3a \sin t \cdot \cos t \, dt \\
 &= \frac{3}{2} a \int_0^{\pi/2} \sin 2t \, dt \\
 &= \frac{3}{4} a [-\cos 2t]_0^{\pi/2}
 \end{aligned}$$

$$\boxed{\therefore S = \frac{3}{2} a} \quad \dots\dots\dots(1)$$

Now the length of the curve ac : Just put $\frac{\pi}{6}$ *insted of* $\frac{\pi}{2}$ because the curve is

Only upto given line.

$$\begin{aligned}
 \therefore S(ac) &= \int_0^{\pi/6} 3a \sin t \cdot \cos t \, dt = \frac{3}{4} a [-\cos 2t]_0^{\pi/6} \\
 &= \frac{3}{4} a \left[-\frac{1}{2} + 1\right]
 \end{aligned}$$

$$\boxed{S(ac) = \frac{3}{8} a} \quad \dots\dots\dots(2)$$

$$\text{Length of remaining part} = \frac{3}{2} a - \frac{3}{8} a = \frac{9}{8} a \quad \dots\dots\dots(3)$$

Divide eqn (3) and (2).

The line $\frac{\pi}{6}$ cuts the given astroid in the ratio of 1:3

Hence proved.

(b) Solve $(D^2 - 7D - 6)y = (1 + x^2)e^{2x}$

[6]

Ans : $(D^2 - 7D - 6)y = (1 + x^2)e^{2x}$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 - 7D - 6) = 0$$

$$\text{Roots are : } D = \frac{7}{2} + \frac{\sqrt{73}}{2}, \frac{7}{2} - \frac{\sqrt{73}}{2}$$

Roots of the given diff. eqn are irrational roots .

$$y_c = e^{\frac{7x}{2}} \left(c_1 \cosh \frac{\sqrt{73}}{2} + c_2 \sinh \frac{\sqrt{73}}{2} \right)$$

For particular integral,

$$\begin{aligned} y_p &= \frac{1}{f(D)} X \\ &= \frac{1}{(D^2 - 7D - 6)} [e^{2x} + e^{2x} x^2] \\ &= \frac{1}{(D^2 - 7D - 6)} e^{2x} + \frac{1}{(D^2 - 7D - 6)} e^{2x} x^2 \\ &= -\frac{e^{2x}}{16} + e^{2x} \frac{1}{(D+2)^2 - 7(D+2) - 6} x^2 \\ &= -\frac{e^{2x}}{16} + e^{2x} \frac{1}{D^2 - 3D - 16} x^2 \\ &= -\frac{e^{2x}}{16} + e^{2x} \left[\frac{1}{-16} \left(\frac{1}{3D - D^2} \right) \right] x^2 \\ &= -\frac{e^{2x}}{16} + e^{2x} \left[\frac{1}{-16} \left(\frac{1}{3D - D^2} \right) \right] x^2 \\ &= -\frac{e^{2x}}{16} \left[1 + \left(1 + \frac{3D - D^2}{16} \right)^{-1} x^2 \right] \\ &= -\frac{e^{2x}}{16} \left\{ 1 + \left[1 - \frac{3D - D^2}{16} + \left(\frac{3D - D^2}{16} \right)^2 \right] x^2 \right\} \\ &= -\frac{e^{2x}}{16} \left\{ 1 + \left[x^2 - \frac{3}{8}x + \frac{2}{16} + \frac{9}{16 \times 8} \right] \right\} \\ &= -\frac{e^{2x}}{16} \left\{ 1 + \left[x^2 - \frac{3}{8}x + \frac{25}{128} \right] \right\} \end{aligned}$$

$$y_p = -\frac{e^{2x}}{16} - \frac{e^{2x}}{16} \left[x^2 - \frac{3}{8}x + \frac{25}{128} \right]$$

The general solution of given diff. eqn is given by ,

$$y_g = y_c + y_p = e^{\frac{7x}{2}} \left(c_1 \cosh \frac{\sqrt{73}}{2} + c_2 \sinh \frac{\sqrt{73}}{2} \right) - \frac{e^{2x}}{16} - \frac{e^{2x}}{16} \left[x^2 - \frac{3}{8}x + \frac{25}{128} \right]$$

(c) Apply Rungee Kutta method of fourth order to find an approximate

Value of y when $x=0.4$ given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y = 1$ when $x = 0$

Taking $h=0.2$.

[8]

Ans : (I) $\frac{dy}{dx} = \frac{y-x}{y+x}$ $x_0 = 0, y_0 = 1, h = 0.2$

$$f(x, y) = \frac{y-x}{y+x}$$

$$k_1 = h \cdot f(x_0, y_0) = 0.2 f(0, 1) = 0.2$$

$$k_2 = h \cdot f \left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2} \right) = 0.2 \cdot f(0.1, 1.1) = 0.1666$$

$$k_3 = h \cdot f \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2} \right) = 0.2 \cdot f(0.1, 1.0833) = 0.1661$$

$$k_4 = h \cdot f(x_0 + h, y_0 + k_3) = 0.2 f(0.2, 1.1661) = 0.1414$$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \frac{0.2 + 2(0.1666) + 2(0.1661) + 0.1414}{6} = 0.1678$$

$$\therefore y(0.2) = y_0 + k = 1 + 0.1678 = 1.1678$$

(II) $x_1 = 0.2, y_2 = 1.1678, h = 0.2$

$$k_5 = h \cdot f(x_1, y_1) = 0.2 f(0.2, 1.1678) = 0.1415$$

$$k_6 = h \cdot f \left(x_1 + \frac{h}{2}, y_1 + \frac{k_5}{2} \right) = 0.2 \cdot f(0.3, 1.23855) = 0.1220$$

$$k_7 = h \cdot f \left(x_1 + \frac{h}{2}, y_1 + \frac{k_6}{2} \right) = 0.2 \cdot f(0.3, 1.2285) = 0.1214$$

$$k_8 = h \cdot f(x_1 + h, y_1 + k_7) = 0.2 f(0.4, 1.2892) = 0.1052$$

$$k^* = \frac{k_5 + 2k_6 + 2k_7 + k_8}{6} = \frac{0.1415 + 2(0.1220) + 2(0.1215) + 0.1052}{6} = 0.1222$$

$$y(0.4) = y_1 + k * = 1.1678 + 0.1222 = 1.290$$

Q.5(a) Use Taylor series method to find a solution of $\frac{dy}{dx} = xy + 1$, $y(0) = 0$
 $X=0.2$ taking $h=0.1$ correct upto 4 decimal places. [6]

Ans: (I) $\frac{dy}{dx} = xy + 1$, $x_0 = 0, y_0 = 0, h=0.1$

$$f(x, y) = 1 + xy$$

$$y' = 1 + xy \qquad y'_0 = 1$$

$$y'' = xy' + y \qquad y''_0 = 0$$

$$y''' = xy'' + 2y' \qquad y'''_0 = 2$$

Taylor's series is given by ,

$$y(0.1) = y_0 + h \cdot y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$$

$$= 0 + 0.1(1) + 0 + \frac{(0.1)^3}{6} (2)$$

$$y(0.1) = 0.1003$$

(II) $x_1 = 0.1, y_1 = 0.1003, h=0.1$

$$y' = 1 + xy \qquad y'_0 = 1.01003$$

$$y'' = xy' + y \qquad y''_0 = 0.201303$$

$$y''' = xy'' + 2y' \qquad y'''_0 = 2.0401903$$

$$\therefore y(0.2) = 0.1003 + 1.01003(0.1) + \frac{0.1^2}{2!} (0.201303) + \frac{0.1^3}{6} (2.0401903)$$

$$\therefore y(0.2) = 0.202708$$

(b) Solve by variation of parameters $(\frac{d^2y}{dx^2} + 1) y = \frac{1}{1+\sin x}$ [6]

Ans : put $\frac{d}{dx} = D$

$$(D^2 + 1)y = \frac{1}{1 + \sin x}$$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 + 1) = 0$$

Roots are : $D = i, -i$

Roots of given diff. eqn are complex.

The complementary solution of given diff. eqn is given by,

$$\therefore y_c = c_1 \cos x + c_2 \sin x$$

For particular solution ,

By method of variation of parameters,

$$y_p = y_1 p_1 + y_2 p_2 \quad \text{where } p_1 = \int \frac{-y_2 X}{w} dx$$

$$p_2 = \int \frac{y_1 X}{w} dx$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$p_1 = \int \frac{-\sin x}{1} dx = \int -\sin x dx = \cos x$$
$$= -\int (\sec x \cdot \tan x - \tan^2 x) dx$$
$$= -[\sec x - \tan x + x]$$

$$p_2 = \int \frac{\cos x}{1} dx = \int \frac{\cos x}{1 + \sin x} dx = \log(1 + \sin x)$$

$$y_p = -[\sec x - \tan x + x] \cos x + \log(1 + \sin x) \sin x$$

The general solution of given diff. eqn is given by ,

$$y_g = y_c + y_p = c_1 \cos x + c_2 \sin x - [\sec x - \tan x + x] \cos x + \log(1 + \sin x) \sin x$$

(c) Compute the value of $\int_{0.2}^{1.4} (\sin x - \ln x + e^x) dx$ using (i) Trapezoidal Rule (ii) Simpson's (1/3)rd rule (iii) Simpson's (3/8)th rule by dividing into six subintervals. [8]

Ans: let $I = \int_{0.2}^{1.4} (\sin x - \ln x + e^x) dx$

Dividing limits in six subintervals .

$$\therefore n = 6 \quad \therefore h = \frac{b-a}{n} = \frac{1.4-0.2}{6} = \frac{1}{5}$$

$x_0 = 0.2$	$x_1 = 0.4$	$x_2 = 0.6$	$x_3 = 0.8$	$x_4 = 1.0$	$x_5 = 1.2$	$x_6 = 1.4$
$y_0 = 3.02$	$y_1 = 2.79$	$y_2 = 2.89$	$y_3 = 3.16$	$y_4 = 3.55$	$y_5 = 4.06$	$y_6 = 4.4$

(i) Trapezoidal rule : $I = \frac{h}{2} [X + 2R]$ -----(1)

$X = \text{sum of extreme ordinates} = 7.42$

$R = \text{sum of remaining ordinates} = 16.45$

$I = \frac{1}{5 \times 2} (7.42 + 2(16.45))$ (from 1)

$I = 4.032$

(ii) Simpson's (1/3)rd rule :

$I = \frac{h}{3} [X + 2E + 4O]$ -----(2)

$X = \text{sum of extreme ordinates} = y_0 + y_6 = 4.4 + 3.02 = 7.42$

$E = \text{sum of even base ordinates} = y_2 + y_4 = 6.44$

$O = \text{sum of odd base ordinates} = y_1 + y_3 + y_5 = 10.01$

$I = \frac{1}{3 \times 5} (7.42 + 2 \times 6.44 + 4 \times 10.01)$ (from 2)

$$I = 4.022$$

(iii) Simpson's $(3/8)^{th}$ rule :

$$I = \frac{3h}{8} [X + 2T + 3R] \quad \text{-----(3)}$$

$$X = \text{sum of extreme ordinates} = y_0 + y_6 = 4.4 + 3.02 = 7.42$$

$$T = \text{sum of multiple of three base ordinates} = y_3 = 3.16$$

$$R = \text{sum of remaining ordinates} = y_1 + y_2 + y_4 + y_5 = 13.49$$

$$\therefore I = \frac{3 \times 1}{8 \times 5} [7.42 + 2 \times 3.16 + 3 \times 13.49]$$

$$\therefore I = 4.02075$$

Q.6(a). Using beta functions evaluate $\int_0^{\pi/6} \cos^6 3\theta \cdot \sin^2 6\theta d\theta$ [6]

Ans: let $I = \int_0^{\pi/6} \cos^6 3\theta \cdot \sin^2 6\theta d\theta$

Put $3\theta = t$

Diff. w.r.t θ ,

$$d\theta = \frac{dt}{3} \quad \text{limits : } [0, \frac{\pi}{2}]$$

$$\therefore I = \frac{1}{3} \int_0^{\pi/2} \cos^6 t \cdot \sin^2 2t dt$$

$$= \frac{4}{3} \int_0^{\pi/2} \cos^3 t (\sin t \cdot \cos t)^2 dt$$

$$= \frac{4}{3} \int_0^{\pi/2} \cos^5 t \cdot \sin^2 t \cdot dt$$

$$= \frac{4}{3} \times \frac{1}{2} \times \beta\left(3, \frac{3}{2}\right) \quad \dots \left\{ \int_0^{\pi/2} \cos^m t \cdot \sin^n t \cdot dt = \frac{1}{2} \times \beta(m+1, n+1) \right\}$$

$$\therefore I = \frac{32}{315}$$

(b) Evaluate $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log(x^2 + y^2) dx dy$ by changing to polar

Coordinates.

[6]

Ans: let $I = \int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log(x^2 + y^2) dx dy$

Region of integration : $y \leq x \leq \sqrt{a^2 - y^2}$

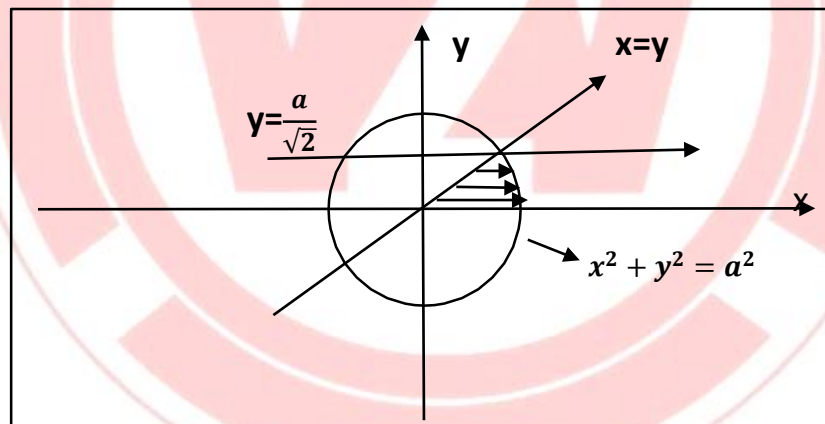
$$0 \leq y \leq \frac{a}{\sqrt{2}}$$

The line $x=y$ is inclined at 45° to the +ve x-axis.

Curves : (i) $x=y$, $y=0$, $y=\frac{a}{\sqrt{2}}$ lines

(ii) $x = \sqrt{a^2 - y^2}$

$x^2 + y^2 = a^2$ circle with centre (0,0) and radius a.



Cartesian coordinates \longrightarrow Polar coordinates

$(x,y) \longrightarrow (r,\theta)$

Put $x = r \cos \theta$ and $y = r \sin \theta$

$$f(x,y) = \log(x^2 + y^2) = \log r^2 = 2 \log r = f(r,\theta)$$

Limits changes to : $0 \leq r \leq a$

$$0 \leq \theta \leq \frac{\pi}{4}$$

$$\begin{aligned} \therefore I &= \int_0^{\frac{\pi}{4}} \int_0^a 2 \log r \cdot r \cdot r \, dr \, d\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \left[\log r \cdot \frac{r^2}{2} - \frac{r^2}{4} \right]_0^a \, d\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \left[\log a \cdot \frac{a^2}{2} - \frac{a^2}{4} \right] \, d\theta \end{aligned}$$

$$\therefore I = \left[\log a \cdot \frac{a^2}{2} - \frac{a^2}{4} \right] \times \frac{\pi}{4}$$

(c) Evaluate $\int \int \int x^2 y z \, dx \, dy \, dz$ over the volume bounded by planes

$$x=0, y=0, z=0 \text{ and } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

[8]

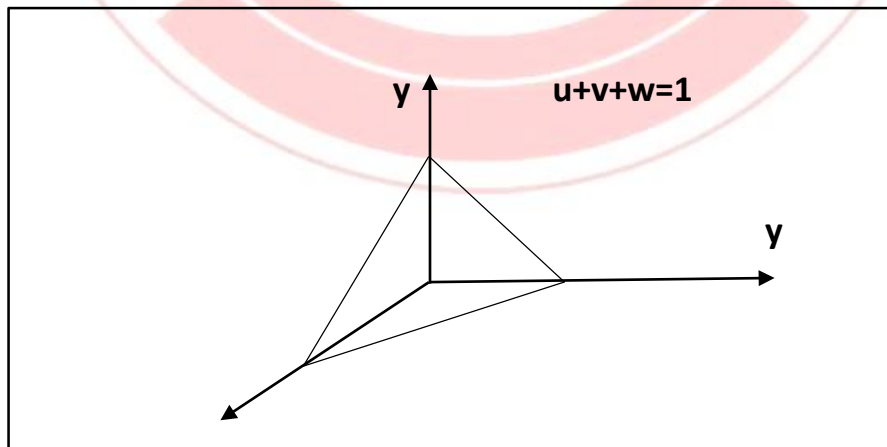
Ans : Let $V = \int \int \int x^2 \, dx \, dy \, dz$

Region of integration is volume bounded by the planes $x=0, y=0, z=0$

And $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Put $x = au$, $y = bv$, $z = cw$

$$\therefore dx \, dy \, dz = abc \, du \, dv \, dw$$



The intersection of tetrahedron with all axes is : $(1,0,0),(0,1,0),(0,0,1)$.

$$0 \leq w \leq (1 - u - v)$$

$$0 \leq v \leq (1 - u)$$

$$0 \leq u \leq 1$$

The volume required is given by ,

$$\begin{aligned} V &= \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc a^2 u^2 b v \cdot c w \cdot du dv dw \\ &= \frac{1}{2} a^3 b^2 c^2 \int_0^1 \int_0^{1-u} u^2 v (1 - u - v)^2 dv du \\ &= \frac{1}{2} a^3 b^2 c^2 \int_0^1 \int_0^{1-u} u^2 v [(1 - u)^2 - 2(1 - u)v + v^2] du dv \\ &= \frac{1}{2} a^3 b^2 c^2 \int_0^1 u^2 \left[(1 - u)^2 \frac{v^2}{2} - 2(1 - u) \frac{v^3}{3} + \frac{v^4}{4} \right]_{v=0}^{v=1-u} du \\ &= \frac{a^3 b^2 c^2}{2} \int_0^1 u^2 (1-u)^4 du \\ &= \frac{a^3 b^2 c^2}{24} \beta(3, 5) \\ &= \frac{a^3 b^2 c^2}{24} \frac{(2!4!)}{7!} \end{aligned}$$

$$\therefore I = \frac{a^3 b^2 c^2}{2520}$$